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E_8 lattice with icosians and Z_5 symmetry

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Abstract. A simple method for the construction of the E_8 root system with icosians is suggested. It is confronted with the root system of E_8 obtained by integral octonions. Embeddings of the maximal subgroups $SU(5) \times SU(5)$ and $SU(3) \times E_6$ in E_8 with respective five-fold and three-fold symmetries are discussed.

1. Introduction

Most general string theories can be obtained using self-dual lattices where the compactified degrees of freedom are described by simple conformal field theories [1]. Among these theories the heterotic string [2] constructed in ten-dimensional spacetime displays most attractive features with a possibility of obtaining the standard model with orbifold compactification [3]. With regard to these facts the root system of E_8 plays an essential role. In a recent paper [4] we have constructed the root system of E_8 with integral octonions [5] and investigated its algebraic structure. The method which we have employed is based on the idea that the octonions are obtained by pairing two quaternions. For this purpose we first obtained the root system of F_4 and then combined two such systems to construct the octonionic description of the E_8 lattice, where one set of F_4 roots is multiplied by an imaginary unit e_7 and added to the other.

In this paper we construct the E_8 lattice with icosians. Icosian is a generic name for the 120 quaternionic elements (q) of the binary icosahedral group [6] which we will discuss in § 2. We follow the same method as [4], i.e. we combine two sets of quaternionic roots of F_4, multiplying one set by $\sigma = \frac{1}{2}(1-\sqrt{5})$ and add it to the other set, which leads to 240 non-zero roots q, σq of E_8. We compare the E_8 roots of icosians with those of octonions and find the relations between them. In § 3 we decompose the roots of E_8 with respect to one of its maximal subgroups $SU(5) \times SU(5)'$ where a five-fold symmetry of icosians plays a dominant role. In § 4 we concentrate on the three-fold symmetry of icosians by branching E_8 with respect to its maximal subgroup $SU(3) \times E_6$. Section 5 consists of the discussions and suggestions as to how this method can be generalised for the construction of the $E_8 \times E_8'$ lattice with the inclusion of octonions and the Leech lattice [7]. A preliminary version of this work has been published [8].

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2. Icosians as roots of E_8

A discrete subgroup of SO(3) of order 60 is called the icosahedral group, which is the symmetry group of the icosahedron. It is isomorphic to the group of even permutations A_5 of five letters. Its double cover $2A_5$ of 120 elements, called the binary icosahedral group, can be represented by quaternions or equivalently 2×2 unitary matrices of determinant one. The 120 elements of the binary icosahedral group can be generated by the elements [6]

$$A=\tfrac{1}{2}(\tau-\sigma e_1+e_3) \qquad B=\tfrac{1}{2}(1-\sigma e_2+\tau e_3) \tag{1}$$

satisfying the relations

$$A^5=B^3=C^2=ABC=-1 \tag{2}$$

with $C=e_3$. Here τ and σ are defined by

$$\begin{aligned} &\tau=\tfrac{1}{2}(1+\sqrt{5}) \qquad \sigma=\tfrac{1}{2}(1-\sqrt{5}) \\ &\tau+\sigma=1 \qquad \tau^2=\tau+1 \qquad \sigma^2=\sigma+1 \qquad \tau\sigma=-1 \end{aligned} \tag{3}$$

and e_1, e_2 and e_3 are the quaternionic imaginary units satisfying

$$e_ie_j=-\delta_{ij}+\varepsilon_{ijk}e_k \qquad \bar{e}_i=-e_i \qquad i,j,k=1,2,3 \tag{4}$$

where δ_{ij} and ε_{ijk} are the usual Kronecker and Levi-Civita symbols respectively. In mathematical literature a short notation $\langle p, q, r\rangle$ is used to denote the groups generated by

$$A^p=B^q=C^r=ABC=-1. \tag{5}$$

Groups generated by quaternions fall into four classes called the quaternion group $\langle 2,2,2\rangle$ of order 8, the binary tetrahedral group $\langle 3,3,2\rangle$ of order 24, the binary octahedral group $\langle 4,3,2\rangle$ of order 48 and finally the binary icosahedral group $\langle 5,3,2\rangle$ of order 120. In [4] we have shown the relations of $\langle 3,3,2\rangle$ and $\langle 4,3,2\rangle$ with the quaternionic root systems of SO(8) and F_4 respectively. An explicit form for the elements of $\langle 5,3,2\rangle$ can be calculated using (1) and is written as follows:

$$\pm 1 \quad \pm e_1 \quad \pm e_2 \quad \pm e_3 \quad \tfrac{1}{2}(\pm 1\pm e_1\pm e_2\pm e_3) \tag{6a}$$

$$\tfrac{1}{2}(\pm\tau\pm e_1\pm\sigma e_2) \quad \tfrac{1}{2}(\pm 1\pm\tau e_1\pm\sigma e_3) \quad \tfrac{1}{2}(\pm\sigma\pm\tau e_2\pm e_3) \quad \tfrac{1}{2}(\pm\sigma e_1\pm e_2\pm\tau e_3) \tag{6b}$$

$$\tfrac{1}{2}(\pm\tau\pm e_2\pm\sigma e_3) \quad \tfrac{1}{2}(\pm 1\pm\sigma e_1\pm\tau e_2) \quad \tfrac{1}{2}(\pm\sigma\pm e_1\pm\tau e_3) \quad \tfrac{1}{2}(\pm\tau e_1\pm\sigma e_2\pm e_3) \tag{6c}$$

$$\tfrac{1}{2}(\pm\tau\pm\sigma e_1\pm e_3) \quad \tfrac{1}{2}(\pm 1\pm\sigma e_2\pm\tau e_3) \quad \tfrac{1}{2}(\pm\sigma\pm\tau e_1\pm e_2) \quad \tfrac{1}{2}(\pm e_1\pm\tau e_2\pm\sigma e_3). \tag{6d}$$

The 24 integral quaternions in (6a) (Hurwitz integers) [9] are the elements of $\langle 3,3,2\rangle$, a subgroup of $\langle 5,3,2\rangle$. $\langle 4,3,2\rangle$ is not a subgroup of $\langle 5,3,2\rangle$. Notice that (6c) and (6d) follow from (6b) by a cyclic permutation of e_1, e_2 and e_3. If we denote by q any element of $\langle 5,3,2\rangle$ then there are 30 elements satisfying $q^2=-1$, 40 elements with $q^3=\pm 1$ (half with $q^3=+1$) and 48 elements with $q^5=\pm 1$ (half with $q^5=+1$). As we shall discuss in §§ 3 and 4, these features of icosians are appropriate for the decompositions of E_8 with respect to its maximal subgroups $SU(3)\times E_6$ and $SU(5)\times SU(5)'$.

Wilson [10], as well as Conway and Sloane [11] have proposed that the E_8 lattice can be described by 120 icosians q and their multiples with σ, σq. To ensure the 'correct angles' between the E_8 roots, they suggested a 'reduced' scalar product. Let p and q be two quaternions. The usual scalar product is defined by

$$(p,q)=\tfrac{1}{2}(\bar{p}q+\bar{q}p). \tag{7}$$

With this definition, the scalar products of icosians (6a)-(6d) will take the values $a+b\sigma$, where a and b are 0, $\pm\frac{1}{2}$, ± 1. The 'reduced' scalar product is defined by the mapping $a+b\sigma \to a$. This new definition of the scalar product also leads to a construction of the Leech lattice in terms of icosians [12]. Therefore, multiplying the elements in (6a)-(6d) by σ will help us to write the complete root system of E_8 explicitly. However, in this section, without referring to the defining relation of (2) we give an alternative construction of the roots of E_8 with icosians. We follow a method similar to that suggested in [4]. We notice that the 'reduced' scalar product allows us to treat σ, σe_1, σe_2, σe_3 as new orthogonal units, independent of the quaternionic units 1, e_1, e_2 and e_3. Thus, by this trick, we enlarge the four-dimensional Euclidean space of quaternions to eight-dimensional Euclidean space, a necessary step towards the construction of the E_8 lattice. This procedure immediately suggests that there must exist a natural correspondence between σ, σe_1, σe_2, σe_3 and the octonionic units e_4, e_5, e_6 and e_7. This point will be clarified in what follows.

We briefly recall the octonionic construction of E_8 in [4]. The root system of the exceptional group F_4 can be written in terms of quaternions as follows

A_0	A_1	A_2	A_3	
24: $\pm 1, \pm e_1, \pm e_2, \pm e_3$ $\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$	8_v: $\frac{1}{2}(\pm 1 \pm e_1)$ $\frac{1}{2}(\pm e_2 \pm e_3)$	8_c: $\frac{1}{2}(\pm 1 \pm e_2)$ $\frac{1}{2}(\pm e_3 \pm e_1)$	8_s: $\frac{1}{2}(\pm 1 \pm e_3)$ $\frac{1}{2}(\pm e_1 \pm e_2)$.	(8)

Here, provided they are multiplied by $\sqrt{2}$, A_0 represents the long roots of F_4 (the root system of SO(8)) and A_1, A_2 and A_3 constitute the short roots of F_4. By pairing two sets of (8) in a delicate manner we obtain the octonionic representation of E_8 roots:

$$[A_0, 0] = A_0 \qquad [0, A_0] = e_7 A_0 \qquad [A_1, A_1] = A_1 + e_7 A_1$$
$$[A_2, A_3] = A_2 + e_7 A_3 \qquad [A_3, A_2] = A_3 + e_7 A_2 \tag{9}$$

where we introduce the octonionic units $e_7^2 = -1$, $\bar{e}_7 = -e_7$, $e_4 = e_7 e_1$, $e_5 = e_7 e_2$ and $e_6 = e_7 e_3$. In fact, this is an explicit realisation of the E_8 lattice with octonions by pairing two F_4 lattices suggested by Goddard *et al* [13] where E_8 sits at one corner of a magic square. A little effort shows that E_8 can also be constructed by pairing two F_4 roots similarly to (9) but with slightly different partners and replacing e_7 by σ. Indeed, one can easily check that the following pairings:

$$(A_0, 0) = A_0 \qquad (0, A_0) = \sigma A_0 \qquad (A_1, A_2) = A_1 + \sigma A_2$$
$$(A_2, A_3) = A_2 + \sigma A_3 \qquad (A_3, A_1) = A_3 + \sigma A_1 \tag{10}$$

not only reproduce the 120 elements in (6a)-(6d), denoted by q, but also yield the additional roots σq of E_8. The differences between two pairings can be contrasted by comparing (9) and (10). This comparison suggests that a correspondence between the octonionic roots and the icosians can be obtained in the following form

$$\begin{array}{lcl} e_7 A_0 \leftrightarrow \sigma A_0 & & e_7 \leftrightarrow \sigma \\ e_7 A_1 \leftrightarrow \sigma A_2 & & e_7 e_1 = e_4 \leftrightarrow \sigma e_2 \\ e_7 A_2 \leftrightarrow \sigma A_1 & \to & e_7 e_2 = e_5 \leftrightarrow \sigma e_1 \\ e_7 A_3 \leftrightarrow \sigma A_3 & & e_7 e_3 = e_6 \leftrightarrow \sigma e_3 . \end{array} \tag{11}$$

With the obvious mapping $1 \leftrightarrow 1$, $e_1 \leftrightarrow e_1$, $e_2 \leftrightarrow e_2$, $e_3 \leftrightarrow e_3$, one can easily transform one system of roots of E_8 into another. This transformation can also be used for the octonionic construction of the Leech lattice, which has been already described by icosians [10-12].

Before we end this section, let us remark on the following facts. There exists an alternative representation of the E_8 lattice with icosians. Instead of starting with the pair (A_1, A_2) in (10), had we started with (A_1, A_3) we would have obtained the following set of icosians:

$$\begin{array}{lll} (A_0, 0) = A_0 & (0, A_0) = \sigma A_0 & (A_1, A_3) = A_1 + \sigma A_3 \\ (A_3, A_2) = A_3 + \sigma A_2 & (A_2, A_1) = A_2 + \sigma A_1 . & \end{array} \tag{12}$$

The 120-element subset in (12), which constitutes the binary icosahedral group, is completely independent of (6*a*)-(6*d*) and can be generated by $A = \frac{1}{2}(\tau + \sigma e_2 + e_3)$ and $B = \frac{1}{2}(1 - \sigma e_1 + \tau e_3)$. Equation (12) is obtained from (10) by a redefinition of the quaternionic units $e_1 \to -e_2$, $e_2 \to e_1$, $e_3 \to e_3$ corresponding to a rotation of $\pi/2$ around the e_3 axis in the clockwise direction, which can be obtained by the action of an element of the octahedral group. Since the octohedral or binary octohedral group is not a subgroup of $\langle 5, 3, 2 \rangle$ the new set of icosians are expected to be different from the former. The elements of the binary icosahedral group used in most of the mathematical literature are those which can be obtained from (12). If one compares (12) and (9), the correspondence in this case between octonions and icosians can be obtained from the mapping

$$\sigma \leftrightarrow e_7 \qquad \sigma e_1 \leftrightarrow e_6 = e_7 e_3 \qquad \sigma e_2 \leftrightarrow e_5 = e_7 e_2 \qquad \sigma e_3 \leftrightarrow e_4 = e_7 e_1 . \tag{13}$$

In the appendix to [4] we have given seven different constructions of the E_8 lattice with octonions similar to (9). Indeed, with the quaternionic units e_1, e_2 and e_3 one can also construct the following two independent octonionic root systems of E_8:

$$\begin{array}{lll} [A_0, 0] = A_0 & [0, A_0] = e_7 A_0 & [A_2, A_2] = A_2 + e_7 A_2 \\ [A_3, A_1] = A_3 + e_7 A_1 & [A_1, A_3] = A_1 + e_7 A_3 & \end{array} \tag{14}$$

and

$$\begin{array}{lll} [A_0, 0] = A_0 & [0, A_0] = e_7 A_0 & [A_3, A_3] = A_3 + e_7 A_3 \\ [A_1, A_2] = A_1 + e_7 A_2 & [A_2, A_1] = A_2 + e_7 A_1 . & \end{array} \tag{15}$$

Relations among these octonionic constructions and those in (10) and (12) can be found in a similar manner which we have illustrated. Although we shall say more in concluding remarks as regards the algebraic aspects of the E_8 roots with octonions contrasted with icosians, some of their properties should be mentioned here since we will use them in later sections. Let P represent an octonionic root of E_8. They satisfy either conditions $q^3 = \pm 1$, $q^2 = \pm 1$. This feature of the octonionic roots can be used for the constructions of orbifolds with Z_6, Z_4, Z_3 and Z_2 symmetries embeddable in E_8 [14]. In the case of icosianic roots we also have the roots satisfying $q^5 = \pm 1$ in addition to the former ones, so that it allows constructions of orbifolds with Z_{10} and Z_5 symmetries.

3. Embedding of $SU(5) \times SU(5)'$ in E_8 with Z_5 symmetry

Hereafter we will deal with the icosian description of the E_8 lattice given by (10). Any element q in (6*b*)-(6*d*) which has the scalar part $-\frac{1}{2}\tau$ or $-\frac{1}{2}\sigma$ has the property $q^5 = 1$.

Therefore 24 elements with this property can be classified into six groups. This classification can be done at will depending on the choices of six elements; to begin with let us choose a root $R_1=\frac{1}{2}(-\tau+e_1+\sigma e_2)$. A simple calculation shows that its powers are given by

$$\begin{aligned} &R_1=\tfrac{1}{2}(-\tau+e_1+\sigma e_2) \qquad R_1^2=\tfrac{1}{2}(-\sigma-\tau e_1+e_2) \\ &R_1^3=\bar{R}_1^2=\tfrac{1}{2}(-\sigma+\tau e_1-e_2) \qquad R_1^4=\bar{R}_1=\tfrac{1}{2}(-\tau-e_1-\sigma e_2) \\ &R_1^5=1 \qquad R_1+R_1^2+R_1^3+R_1^4+R_1^5=0. \end{aligned} \tag{16}$$

One can immediately check that with the use of the 'reduced' scalar product these roots can be used for the description of an extended Coxeter-Dynkin diagram of SU(5) (figure 1). The 20 non-zero roots of SU(5) are then given by

$$\begin{aligned} &\pm R_1, \quad \pm R_1^2, \quad \pm R_1^3, \quad \pm R_1^4, \quad \pm R_1^5=\pm 1 \\ &\pm\sigma R_1, \quad \pm\sigma R_1^2, \quad \pm\sigma R_1^3, \quad \pm\sigma R_1^4, \quad \pm\sigma R_1^5=\pm\sigma. \end{aligned} \tag{17}$$

It is clear that this set of SU(5) roots is left invariant under a repeated left or right multiplication of R_1, leading to a Z_5 symmetry. An orthogonal set of roots to the roots in (17) can be generated by repeated application of R_1 on the roots $\pm e_3$ and $\pm\sigma e_3$. Thus we obtain the roots of another SU(5)′, orthogonal to SU(5), given by

$$\begin{aligned} &\pm R_1 e_3, \quad \pm R_1^2 e_3, \quad \pm R_1^3 e_3, \quad \pm R_1^4 e_3, \quad \pm e_3, \\ &\pm\sigma R_1 e_3, \quad \pm\sigma R_1^2 e_3, \quad \pm\sigma R_1^3 e_3, \quad \pm\sigma R_1^4 e_3, \quad \pm\sigma e_3. \end{aligned} \tag{18}$$

Hence (17) and (18) display the roots of the maximal subgroup SU(5)×SU(5)′ of E_8 with an obvious Z_5 symmetry. The remaining roots belong to the coset space E_8/SU(5)× SU(5)′, which transform as ($\underline{5}$, $\underline{10}$*)+($\underline{5}$*, $\underline{10}$) and ($\underline{10}$, $\underline{5}$)+($\underline{10}$*, $\underline{5}$*). The 200 roots of the coset space can be written in a compact form by defining

$$\begin{aligned} &A_1=\tfrac{1}{2}(-1-e_1-e_2+e_3) \qquad A_2=\tfrac{1}{2}(-\sigma-\tau e_2+e_3) \qquad A_3=\tfrac{1}{2}(-\tau e_1+\sigma e_2+e_3) \\ &A_4=\tfrac{1}{2}(\sigma-\tau e_2+e_3)=-\bar{A}_2 \qquad A_5=\tfrac{1}{2}(1-e_1-e_2+e_3)=-\bar{A}_1 \end{aligned} \tag{19a}$$

$$\begin{aligned} &B_1=\tfrac{1}{2}(\tau-e_2+\sigma e_3) \qquad B_2=\tfrac{1}{2}(-1-\tau e_1+\sigma e_3) \qquad B_3=\tfrac{1}{2}(e_1-\tau e_2+\sigma e_3) \\ &B_4=\tfrac{1}{2}(1-\tau e_1+\sigma e_3)=-\bar{B}_2 \qquad B_5=\tfrac{1}{2}(-\tau-e_2+\sigma e_3)=-\bar{B}_1. \end{aligned} \tag{19b}$$

It can be shown that the roots of the representation ($\underline{5}$, $\underline{10}$*) are given by the elements

$$R_1^n A_m, \quad \sigma R_1^n B_m \qquad n, m=1,2,3,4,5. \tag{20}$$

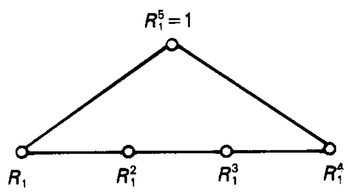

Figure 1. Extended Coxeter-Dynkin diagram of SU(5) with five-fold symmetry of icosian $R_1=\frac{1}{2}(-\tau+e_1+\sigma e_2)$.

Negatives of these roots constitute the conjugate representation ($\underline{5}^*$, $\underline{10}$). Similarly the roots belonging to ($\underline{10}$, $\underline{5}$) can be written as

$$R_1^n B_m, \quad -\sigma R_1^n A_m \qquad n, m = 1, 2, 3, 4, 5 \tag{21}$$

the negatives of which represent the roots in ($\underline{10}^*$, $\underline{5}^*$). It is obvious from (20) and (21) that the roots belonging to the representations ($\underline{5}$, $\underline{10}^*$), ($\underline{10}$, $\underline{5}$) and their conjugates preserve Z_5 symmetry separately.

It is perhaps more convenient to express the roots of the coset space as products of R_1^n with the elements of the binary tetrahedral group which can be defined by

$$S_0 = \tfrac{1}{2}(1 + e_1 + e_2 + e_3) \qquad S_1 = \tfrac{1}{2}(1 + e_1 - e_2 - e_3)$$
$$S_2 = \tfrac{1}{2}(1 - e_1 + e_2 - e_3) \qquad S_3 = \tfrac{1}{2}(1 - e_1 - e_2 + e_3) \tag{22}$$

with their conjugates and negatives. Then all the roots of E_8 can be decomposed as follows:

$$(20, 1): \pm R_1^n, \quad \pm\sigma R_1^n \qquad (1, 20): \pm R_1^n e_3, \quad \pm\sigma R_1^n e_3 \tag{23a}$$

$$(\underline{5}, \underline{10}^*) + (\underline{5}^*, \underline{10}) + (\underline{10}, \underline{5}) + (\underline{10}^*, \underline{5}^*): \pm R_1^n(S_\alpha, \bar{S}_\beta, e_1, e_2), \quad \pm\sigma R_1^n(S_\alpha, \bar{S}_\beta, e_1, e_2)$$
$$n = 1, 2, 3, 4, 5 \qquad \alpha, \beta = 0, 1, 2, 3. \tag{23b}$$

As we stated at the beginning of this section, R_1 can be chosen six different ways:

$$R_1 = \tfrac{1}{2}(-\tau + e_1 + \sigma e_2) \qquad R_2 = \tfrac{1}{2}(-\tau + e_2 + \sigma e_3) \qquad R_3 = \tfrac{1}{2}(-\tau + \sigma e_1 + e_3) \tag{24a}$$

$$R_4 = \tfrac{1}{2}(-\tau + e_1 - \sigma e_2) \qquad R_5 = \tfrac{1}{2}(-\tau + e_2 - \sigma e_3) \qquad R_6 = \tfrac{1}{2}(-\tau - \sigma e_1 + e_3). \tag{24b}$$

These choices are made so that the triples (R_1, R_2, R_3) and (R_4, R_5, R_6) have cyclic symmetries in e_1, e_2 and e_3. It is clear from these discussions that $SU(5) \times SU(5)'$ can be embedded in E_8 in six different ways; in each case a Z_5 symmetry is manifest. These six possible decompositions of E_8 with respect to $SU(5) \times SU(5)'$ can be displayed as follows:

(20, 1)	(1, 20)	($\underline{5}$, $\underline{10}^*$) + ($\underline{5}^*$, $\underline{10}$) + ($\underline{10}$, $\underline{5}$) + ($\underline{10}^*$, $\underline{5}^*$)	
$\pm R_a^n, \pm\sigma R_a^n$	$\pm R_a^n e_3, \pm\sigma R_a^n e_3$	$(\pm R_a^n, \pm\sigma R_a^n)(S_\alpha, \bar{S}_\beta, e_1, e_2)$	(25a)
$\pm R_b^n, \pm\sigma R_b^n$	$\pm R_b^n e_1, \pm\sigma R_b^n e_1$	$(\pm R_b^n, \pm\sigma R_b^n)(S_\alpha, \bar{S}_\beta, e_2, e_3)$	(25b)
$\pm R_c^n, \pm\sigma R_c^n$	$\pm R_c^n e_2, \pm\sigma R_c^n e_2$	$(\pm R_c^n, \pm\sigma R_c^n)(S_\alpha, \bar{S}_\beta, e_3, e_1)$	(25c)

where $n = 1, 2, 3, 4, 5$, $a = 1, 4$, $b = 2, 5$, $c = 3, 6$ and $\alpha, \beta = 0, 1, 3$

So far we have discussed the action of an element of the binary icosahedral group on its elements by left or right multiplication. We can also consider a transformation of an element where a left and right multiplication are combined. Let P and Q be elements of $\langle 5, 3, 2 \rangle$. Then a transformation of the form

$$Q' = (\pm P)Q(\pm\bar{P}) \tag{26}$$

represents the action of an element of the icosahedral group on the elements of $\langle 5, 3, 2 \rangle$. It is also possible to generate a Z_5 symmetry in this form. For an illustration of this point let us choose $P = R_1 = \frac{1}{2}(-\tau + e_1 + \sigma e_2)$. With this choice of P each SU(5) root remains invariant without being affected at all by the group operation. However, the

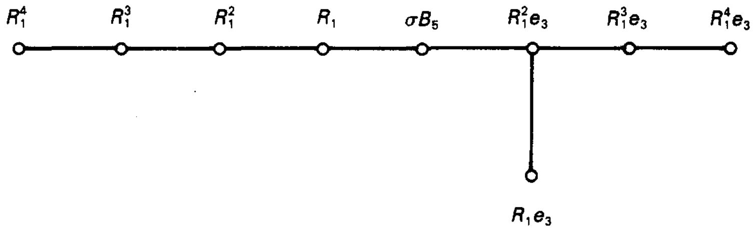

Figure 2. Extended Coxeter-Dynkin diagram of E_8 leading to an $SU(5) \times SU(5)'$ embedding with Z_5 symmetry.

same action (26) on the roots of SU(5)′ manifests itself as a Z_5 symmetry. An extended Coxeter-Dynkin diagram of E_8, where the Z_5 symmetries so far discussed are apparent, is given in figure 2. An SU(5) orbifold with Z_5 symmetry can be used for the construction of six-dimensional string theories [15].

4. Embedding of SU(3)×E_6 in E_8 with Z_3 symmetry

In this section we decompose E_8 roots under its maximal subgroup SU(3) × E_6, which plays an important role in orbifold compactification of the heterotic string. We will choose the simple roots of E_8 in such a way that the extended Coxeter-Dynkin diagram of E_6 obtained from that of E_8 will have a three-fold symmetry of icosians. Such an extended Coxeter-Dynkin diagram of E_6 is given in figure 3. Here Z_3 symmetry is defined in the form of the transformation (26) where P is replaced by $S_0 = \frac{1}{2}(1 + e_1 + e_2 + e_3)$, which satisfies $S_0^2 = -\bar{S}_0$, $S_0^3 = -1$, $S_0 + \bar{S}_0 = 1$. It is easy to show that an action of S_0 in the form of (26) rotates e_1, e_2, e_3 and σe_1, σe_2, σe_3 in cyclic order. We choose the simple roots of E_8 in such a way that the roots of E_6 include the elements of the binary tetrahedral group. SU(3) roots orthogonal to those of E_6 in figure 3 are given by $\pm\sigma$, $\pm\sigma S_0$, $\pm\sigma\bar{S}_0$. Each of these roots is invariant under the action of S_0 in the form of (26). It is clear from figure 3 that S_0 rotates B_1, B_2 and B_3 in cyclic order just like it does e_1, e_2 and e_3. Thus, Z_3 symmetry of the extended Coxeter-Dynkin diagram of E_6 is obtained by repeated application of the element S_0 of the E_8 root

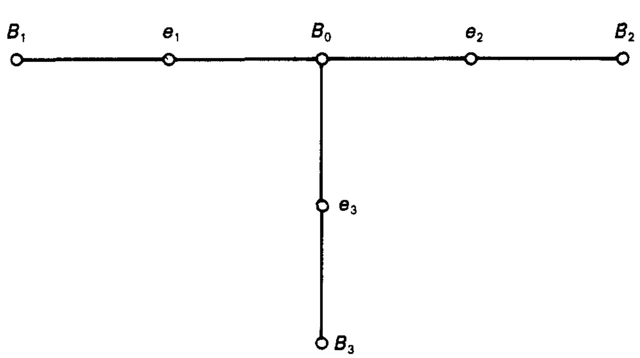

Figure 3. Extended Coxeter-Dynkin diagram of E_6 with icosians $B_1 = \frac{1}{2}(-1 - \tau e_1 - \sigma e_3)$, $B_2 = \frac{1}{2}(-1 - \tau e_2 - \sigma e_1)$, $B_3 = \frac{1}{2}(-1 - \tau e_3 - \sigma e_2)$, $B_0 = \bar{S} = (1 - e_1 - e_2 - e_3)$, e_1, e_2 and e_3.

system. The 72 non-zero roots of E_6 can be grouped as 24 triples, each of which is rotated by the action of S_0. We do not give the details here, but it can be easily shown that while 48 of the roots of E_6 come from the elements of the binary icosahedral group, the remaining roots are the σ multiples of the elements of the icosahedral group lying in the coset space $E_8/SU(3)\times E_6$.

An interesting algebraic property of the roots B_1, B_2 and B_3 is the relations [14]:

$$[B_i, B_j] = -B_0 - \varepsilon_{ijk}B_k \qquad B_iB_jB_k = B_iS_0$$
$$B_0 = \bar{S}_0 \qquad i\neq j\neq k = 1, 2, 3. \tag{27}$$

Similar relations can be obtained for the roots cyclically rotated to each other. Let Q_1, Q_2 and Q_3 be such roots of E_6 belonging to $\langle 5, 3, 2\rangle$. Then we can show that their triple products can be classified as

$$Q_1Q_2Q_3 = \begin{cases} Q_1S_0 & \text{for } (Q_1, \bar{S}_0) = 0 \\ 1 & \text{for } (Q_1, \bar{S}_0) = \frac{1}{2} \\ -1 & \text{for } (Q_1, \bar{S}_0) = -\frac{1}{2}. \end{cases} \tag{28}$$

The scalar products must be understood in the 'reduced' form. A cyclic rotation in (28) is also implicit.

Using the notation of (10), we can give an explicit decomposition of the E_8 roots under $SU(3)\times E_6$:

$$\begin{array}{lllll} (0, & \pm 1 & \pm e_1 & \pm e_2 & \pm e_3) \\ (0, & \pm S_0 & \pm \bar{S}_2 & \pm S_1 & \pm \bar{S}_1) \\ (0, & \pm \bar{S}_0 & \pm S_3 & \pm \bar{S}_3 & \pm S_2) \end{array} \tag{29a}$$

$$\begin{array}{lllll} (A_1, & \pm\frac{1}{2}(e_3-e_1) & \pm\frac{1}{2}(1-e_2) & \pm\frac{1}{2}(e_1+e_3) & \pm\frac{1}{2}(1+e_2)) \\ (A_2, & \pm\frac{1}{2}(e_1-e_2) & \pm\frac{1}{2}(1+e_3) & \pm\frac{1}{2}(1-e_3) & \pm\frac{1}{2}(e_1+e_2)) \\ (A_3, & \pm\frac{1}{2}(e_2-e_3) & \pm\frac{1}{2}(e_3+e_2) & \pm\frac{1}{2}(1+e_1) & \pm\frac{1}{2}(1-e_1)) \\ (A_0, & 0). & & & \end{array} \tag{29b}$$

Our notation needs clarification. A bracket $(a, b\,c\,d\,e)$ represents four brackets of the form $(a, b), (a, c), (a, d), (a, e)$. In this unusual notation $(0, \pm 1) = \pm\sigma$, $(0, \pm S_0) = \pm\sigma S_0$, $(0, \pm\bar{S}_0) = \pm\sigma\bar{S}_0$ are the roots of SU(3). E_6 roots are given in the same column as SU(3) roots in (29) which can be also written as

$$A_0 \qquad A_1 \pm \tfrac{1}{2}\sigma(e_3-e_1) \qquad A_2 \pm \tfrac{1}{2}\sigma(e_1-e_2) \qquad A_3 \pm \tfrac{1}{2}\sigma(e_2-e_3). \tag{30}$$

Since S_0 is an element of A_0 it is left unchanged by the action of S_0. The other sets of elements are rotated into each other in cyclic order since A_1, A_2 and A_3 are rotated into each other by S_0 in the form of (26). It is quite obvious that the remaining elements belong to the coset space $E_8/SU(3)\times E_6$ and are separately rotated into each other by S_0.

The roots of E_8 in (29) are organised such that each column is invariant under right multiplication by S_0, in contrast to the transformation defined by (26). While each column in (29a) represents the roots of SU(3), the column below displays the roots of E_6 provided $(A_0, 0)$ is also added. Therefore, the root system given in (29) shows that $SU(3)\times E_6$ can be embedded in E_8 in four different ways with an obvious S_0 invariance by right multiplication. However, an action of S_0 in the sense of (26) will allow only one decomposition. Similar decompositions of E_8 roots under SU(3) × E_6 can be made, replacing S_0 by any icosian which satisfies $q^3 = \pm 1$.

String phenomenologists sometimes prefer an $SU(3)^3$ orbifold with Z_3 symmetry [3]. The simple roots of E_8 can be arranged so that a Z_3 symmetry for each $SU(3)^4 \subset SU(3) \times E_6 \subset E_8$ can be realised. For this purpose let us choose a root $R = \frac{1}{2}(-1 + \tau e_1 + \sigma e_3)$. Then it can be shown that each of the following sets of roots:

$$\begin{array}{ll} (\pm R, \pm R^2, \pm 1) & (\pm R e_2, \pm R^2 e_2, \pm e_2) \\ (\pm \sigma R, \pm \sigma R^2, \pm \sigma) & (\pm \sigma R e_2, \pm \sigma R^2 e_2, \pm \sigma e_2) \end{array} \tag{31}$$

represents one SU(3). Since $R^3 = 1$, each SU(3) has a Z_3 symmetry. The Z_3 symmetry discussed in this section can also be represented by the octonionic roots of E_8, which will be discussed in a separate publication [16].

5. Discussion and conclusion

Several aspects of the representation of the E_8 lattice with icosians differ from the octonionic representation. The two descriptions of the E_8 lattice can be contrasted as follows.

(i) The octonionic root system obeys the usual scalar product defined by (7) and forms a closed non-associative discrete algebra of order 240, only 24 of which satisfy the group property of the binary tetrahedral group.

(ii) In the case of icosians, the order of the group structure is extended to the binary icosahdral group of order 120 but the whole set of 240 roots do not close under multiplication since a multiplication of the form $(\sigma q)(\sigma q) = \sigma^2 q^2 = q + \sigma q$ produces lattice vectors of higher norms. Icosians represent the E_8 lattice only with the 'reduced' scalar product.

(iii) Octonionic roots yield natural Abelian symmetries Z_6, Z_4, Z_3 and Z_2 of the E_8 lattice with an interesting manifestation of the triality of the extended Coxeter-Dynkin diagram of E_6 [14].

(iv) With icosians, while preserving the three-fold symmetry of the extended Coxeter-Dynkin diagram of E_6, one can naturally extend the Abelian symmetries to Z_{10}, Z_6, Z_5, Z_4, Z_3 and Z_2 of the root system of E_8. To be more specific, the maximal subgroup SU(5) × SU(5)′ can be embedded in E_8 with a Z_5 symmetry invariance, which is not possible in the octonionic representation of the E_8 lattice.

Another amusing observation is the possibility of describing the $E_8 \times E_8'$ root system by a simple extension of the root system given by (10). If we multiply the icosianic roots in (10) by the octonionic imaginary unit e_7 we obtain an independent root system of 240 elements described by the octonionic units e_4, e_5, e_6, e_7 and their σ multiples. Then one can show that the 120 elements of the binary icosahedral group in E_8 and the corresponding 120 octonionic elements in E_8' form a closed algebra, presumably a subset of a larger algebra. The products of any two elements of the E_8' lattice will yield a lattice element of E_8. Another possible description of the $E_8 \times E_8'$ lattice can be made by multiplying the octonionic roots of E_8 by σ. This second construction is totally different from the previous one in most respects.

The first construction of $E_8 \times E_8'$ could be attractive in view of the fact that an unbroken E_8' in the heterotic string could be attributed to its pure octonionic structure.

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